A REMARK ON POSITIVELY CURVED MANIFOLDS OF DIMENSIONS 7 AND 13

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In the present paper we construct totally geodesic embeddings of some 7-dimensional manifolds into 13-dimensional manifolds with positive sectional curvature and explain the strange coincidence of pinching constants of the normally homogeneous Berger space ([Be]) and the homogeneous Aloff-Wallach space $N_{1,1,-1/2}$ ([AW]). This constant is equal to $\frac{16}{29\cdot 37}$ as it was established by Heintze for the Berger space ([H]) and by Huang ([Hu]) for the Aloff-Wallach space. Moreover these totally geodesic embeddings shed light on a relation of the well-known 7-dimensional manifolds constructed by Aloff, Wallach and Eschenburg ([E1]) to the series of 13-dimensional positively curved manifolds founded recently by Bazaikin ([Ba]).

We identify the Lie groups $U(n)$ with the groups formed by $(n\times n)$-matrices A such that

$$A\cdot I_n\cdot A^* = I_n$$

where I_n is the unit $(n\times n)$-matrix and $A^*_{ij}=\bar{A}_{ji}$. We consider the Lie groups $SU(n)$ as the subgroups of $U(n)$ formed by matrices with $\det = 1$. We mean by $Sp(2)$ a subgroup of $SU(4)$ formed by matrices A which satisfy the following equality

$$A\cdot\begin{pmatrix} 0 & I_2\\ -I_2 & 0\end{pmatrix}\cdot A^t = \begin{pmatrix} 0 & I_2\\ -I_2 & 0\end{pmatrix}$$

where $A^t_{ij}=A_{ji}$. Moreover we will consider $Sp(2)$ as a subgroup of $SU(5)$ formed by matrices of the form

$$\begin{pmatrix} A & 0\\ 0 & 1\end{pmatrix}\in SU(5),\quad A\in Sp(2)\subset SU(4).$$

We consider the Lie algebras of these groups as realized by matrices in usual manner.

We denote by T^1, T^1_p and $T_{(0)}$ the subgroups of $U(5)$ formed by diagonal matrices $diag(z,z,z,z,z^{-4})$, $diag(z^p,z^p,z^p,z^p,z)$, and $diag(z,z,z,z,1)$ respectively where $|z|=1$ and p is a positive integer number.

First we remind the known examples (we consider only simply connected manifolds).

1) 7-dimensional manifolds.

1.1) 7-dimensional Berger space ([Be]).

This space is isometric to a factor-space $Sp(2)/SU(2)$ where $Sp(2)$ is endowed with a standard biinvariant metric and an embedding $SU(2)\subset Sp(2)$ is a nonstandard one. We will not discuss this space and consider it as an exceptional one.

1.2) Aloff-Wallach spaces ([AW]).

Let $T_{k,l}$ be the subgroup of $SU(3)$ formed by diagonal matrices $(z^k, z^l, z^{-(k+l)})$. We consider the subgroup $G_1 = U(2)$ of $SU(3)$ given by

$$G_1 = \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}, \quad A\in U(2)$$

and denote by g_1 the Lie algebra of G_1. We denote by $f_{k,l}$ the Lie algebra of $T_{k,l}$. One can see that $f_{k,l}$ is generated by a diagonal matrix $diag(2\pi\sqrt{-1}k, 2\pi\sqrt{-1}l, -2\pi\sqrt{-1}(k+l))$. Let denote by $\langle,\rangle_0$ the Killing biinvariant metric on $SU(3)$. Then one can consider a homogeneous metric, on factor-space $N_{k,l,t} = SU(3)/T_{k,l}$, generated by the metric

$$\langle x, y\rangle = (1+t)\langle x_1, y_1\rangle_0 + \langle x_2, y_2\rangle_0 \tag{1}$$

where $x_i, y_i \in V_i$ and $f^{\perp}_{k,l} = V_1 \oplus V_2$ is an orthogonal decomposition, $V_1 = f^{\perp}_{k,l} \cap g_1$ and $V_2 = g_1^{\perp}$.

Aloff and Wallach had shown that if k and l have the same sign and $-1 < t < 0$ then these manifolds $N_{k,l,t}$ are positively curved.

1.3) Eschenburg spaces ([E1,E2]).

These spaces are generalizations of the previous ones and have form $T_{k,l}\backslash U(3)/T_{p,q}$. For suitable integers k, l, p and q and a metric on $U(3)$ these manifolds would have positive curvature. We will not dwell on them and only notice that most of them are not homeomorphic to homogeneous manifolds and these manifolds were first examples of such kind.

2) 13-dimensional manifolds.

2.1) 13-dimensional Berger space ([Be]).

This manifold B^{13} is a factor-space $SU(5)/(Sp(2)\times T^1)$ where $SU(5)$ is endowed with the Killing biinvariant metric.

2.2) Bazaikin spaces ([Ba]).

These spaces have form $S_{\bar p}\backslash U(5)/(Sp(2)\times T_{(0)})$ where $S_{\bar p}$ is formed by diagonal matrices $diag(z^{p_1}, z^{p_2}, z^{p_3}, z^{p_4}, z^{p_5})$ where $|z| = 1$. For suitable tuples of integers $\bar p$ and a metric on $U(5)$ these manifolds would have positive curvature. We also mention that for these examples a metric on $U(5)$ is taken to be left-invariant under $U(5)$-action and right-invariant under $U(4)\times U(1)$-action where subgroup $U(4)\times U(1)$ formed by diagonal-block matrices with 4×4- and 1×1- blocks.

Now we proceed with our main construction.

Put

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-1} \end{pmatrix}$$

and define the following mapping

$$\sigma : U(5) \to U(5) \quad : \quad A \to S \cdot A \cdot S^{-1}.$$

The following two propositions are evident.

Proposition 1. *$\sigma(G) = G$ for $G = SU(5), Sp(2), T^1, T^1_p, T_{(0)}$.*

Proposition 2. *σ^2 is an identical mapping, i.e., σ is an involution.*

Since metric on $SU(5)$ is biinvariant for the Berger space and the metric on is $U(5)$-left-invariant and $U(4)\times U(1)$-right-invariant for the Bazaikin spaces $T^1_p\backslash U(5)/(Sp(2)\times T_{(0)})$, the following Proposition holds.

Proposition 3. *The involution σ induces isometric involutions on the spaces B^{13} and $T^1_p\backslash U(5)/(Sp(2)\times T_{(0)})(p > 0)$.*

First we consider the action of this involution on the Berger space B^{13}.

Theorem 1. *Let W^7 be a submanifold of B^{13} which contains the point $E = 1\cdot(Sp(2)\times T^1) \in B^{13}$ where 1 is the unit of $SU(5)$ and which is formed by fixed points of involution $\sigma : B^{13} \to B^{13}$. Then the manifold W^7 is a totally geodesic submanifold which is isometric to the Aloff-Wallach space $N_{1,1,-1/2}$ and minimal and maximal values of the sectional curvature of B^{13} are attained on two-planes tangent to W^7.*

Proof of Theorem 1.

First notice that, since W^7 is a component of the set formed by fixed points of involution, this embedding $W^7 \to B^{13}$ is a totally geodesic one.

Now let compute the dimension of W^7 and find generators of the tangent space of W^7 at the point E.

We use notations from the paper of Heintze ([H]) who denoted by $H_i (1 \le i \le 11)$ a set of orthonormal vectors which form a basis for tangent space to $Sp(2)\times T^1$ and denoted by $M_j (1 \le j \le 13)$ basic orthonormal vectors of its orthogonal complement.

Let E_{kl} be a (5×5)-matrix $(\delta_{ak}\delta_{bl})_{(1\le a,b\le 5)}$. Then let introduce $Q_{kl} = E_{kl} - E_{lk}$, $R_{kl} = \sqrt{-1}(E_{kl} + E_{lk})$ and $P_k = \sqrt{-1}(E_{kk} - E_{55})$.

Heintze used the following basis :

$$M_j = \sqrt{2}Q_{j5}, \quad M_{j+4} = \sqrt{2}R_{j5}, \quad j = 1, 2, 3, 4,$$

$$M_9 = Q_{12} - Q_{34}, \quad M_{10} = Q_{14} - Q_{23},$$

$$M_{11} = R_{12} + R_{34}, \quad M_{12} = R_{14} - R_{23}, \quad M_{13} = P_1 - P_2 + P_3 - P_4.$$

One can derive by direct computations that the space V generated by these vectors splits into two pairwise orthogonal subspaces V^+ and V^- such that $\sigma|_{V^\pm} = \pm 1$.

The orthonormal bases of these subspaces are :

1) for V^+ :

$$\frac{M_1+M_7}{\sqrt{2}}, \frac{M_2-M_8}{\sqrt{2}}, \frac{M_3-M_5}{\sqrt{2}}, \frac{M_4-M_6}{\sqrt{2}}, M_{11}, M_{12}, M_{13};$$

2) for V^- :

$$\frac{M_1-M_7}{\sqrt{2}}, \frac{M_2-M_8}{\sqrt{2}}, \frac{M_3+M_5}{\sqrt{2}}, \frac{M_4+M_6}{\sqrt{2}}, M_9, M_{10}.$$

Since $\dim V^+ = 7$ and the submanifold W^7 is homogeneous,

$$\dim W^7 = 7.$$

Let show that this submanifold is isometric to the space $N_{1,1,-1/2}$.

We introduce another action given by

$$\rho : SU(5) \to SU(5) \quad : \quad A \to R \cdot A \cdot R^{-1}$$

where

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 & \sqrt{-1} & 0 \\ \sqrt{-1} & 0 & 1 & 0 & 0 \\ 0 & \sqrt{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

Since $R \in Sp(2) \subset SU(5)$, this action is an isometry.

The action ρ generates an action on the Lie algebra $su(5)$ which we denote also by ρ. We compute the action of ρ only on V^+ :

$$\rho(\frac{M_1+M_7}{\sqrt{2}}) = M_7, \quad \rho(\frac{M_2+M_8}{\sqrt{2}}) = M_8,$$

$$\rho(\frac{M_3-M_5}{\sqrt{2}}) = M_3, \quad \rho(\frac{M_4-M_6}{\sqrt{2}}) = M_4, \tag{2}$$

$$\rho(M_{11}) = M_{11}, \quad \rho(M_{12}) = M_9, \quad \rho(M_{13}) = M_{13}.$$

Moreover we have

$$M_9 = -2Q_{34} + \rho(H_{10}),\ M_{11} = 2R_{34} + \rho(H_8),\ M_{13} = 2(P_3 - P_4) + \rho(H_5) \tag{3}$$

where H_j are unit basic orthonormal vectors from the tangent space to $Sp(2) \times T^1$ (see [H]).

One can see that $(M_3, M_4, M_7, M_8, \sqrt{2}Q_{34}, \sqrt{2}R_{34}, \sqrt{2}(P_3 - P_4))$ form an orthonormal (up to multiplication by constant) basis in a subalgebra V_1 (see (1)) for $k = l = 1$ and the group $SU(3)$ given by

$$\tilde{G} = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}, \qquad A \in SU(3). \tag{4}$$

But , since (2) and (3), we have that $(M_3, M_4, M_7, M_8, 2Q_{34}, 2R_{34}, 2(P_3 - P_4))$ form an orthonormal basis in the tangent space of $\rho(W)$ at the point E. This coincides with (1) for $t = -\frac{1}{2}$.

Since W^7 is a homogeneous manifold and the homogeneous manifold $N_{1,1,-1/2}$ is simply connected, there exists a finite isometric covering

$$N_{1,1,-1/2} \to W^7.$$

Let prove that this covering is a diffeomorphism.

From formulas (2) and (3) one can see that W^7 is formed by $Sp(2) \times T^1$-orbits of elements g such that $\rho(g) \in \tilde{G}$ (see (4)). But by direct computations one can derive that

1) if $\rho(h) \in \tilde{G}$ and $h \in Sp(2)$ then $h = 1 \in Sp(2)$;

2) if

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$$

and $\rho(h)$ is a diagonal matrix then $a = d, b = -c, a^2 + b^2 = I_2$ and $\rho(h)$ has the form $diag(\lambda, \mu, \lambda^{-1}, \mu^{-1}, 1)$.

It follows now that orbits of two elements $g_1, g_2 \in \tilde{G}$ coincide if and only if $g_1 g_2^{-1} \in T_{1,1} \subset SU(3)$ and we conclude that

$$W^7 = N_{1,1,-1/2}.$$

We are left to prove that the curvature of B^{13} attains its minimal and maximal values on two-planes which are tangent to W^7.

By using of the formula from Lemma 2 of [H] one compute the sectional curvature of 2-plane generated by vectors X and Y :

$$K(X,Y) = \frac{29}{4}$$

for $X = \frac{M_1 + M_7}{\sqrt{2}}, Y = \frac{M_3 - M_5}{\sqrt{2}}$ and it was proved in [H] that this value is the maximum of curvature of the space B^{13}.

Let take a matrix $Q = diag(1, -\sqrt{-1}, 1, \sqrt{-1}, 1) \in Sp(2) \subset SU(5)$. Since $Q \in Sp(2)$, the action $\xi : X \to Q \cdot X \cdot Q^{-1}$ generates an isometry of B^{13} (see [H]). Let take

$$X = \sqrt{\frac{12}{37}}\left(\frac{M_1 + M_7}{\sqrt{2}} + \frac{M_2 + M_8}{\sqrt{2}}\right) + \sqrt{\frac{13}{37}} M_{11},$$

$$Y = -\sqrt{\frac{12}{37}}\left(\frac{M_3 - M_5}{\sqrt{2}} - \frac{M_4 - M_6}{\sqrt{2}}\right) - \sqrt{\frac{13}{37}}M_{12}.$$

One can compute

$$\xi^{-1}(X) = \sqrt{\frac{12}{37}}\left(\frac{M_1 + M_7}{\sqrt{2}} + \frac{M_4 + M_6}{\sqrt{2}}\right) + \sqrt{\frac{13}{37}}M_9,$$

$$\xi^{-1}(Y) = -\sqrt{\frac{12}{37}}\left(\frac{M_3 - M_5}{\sqrt{2}} - \frac{M_2 - M_8}{\sqrt{2}}\right) + \sqrt{\frac{13}{37}}M_{10},$$

apply to these vectors Lemma 2 of [H] and derive that

$$K(X,Y) = K(\xi^{-1}(X), \xi^{-1}(Y)) = \frac{4}{37}.$$

It was proved in [H] that this value is the minimum of the curvature of B^{13}.

One can conclude that the pinching constants $K_{\min}/K_{\max}$ for B^{13} and $W^7 = N_{1,1,-1/2}$ coincide and are equal to $16/29 \cdot 37$.

Theorem 1 is established.

Of course, this explanation can be simplified by replacing the involution σ by another one:

$$A \to \Sigma \cdot A \cdot \Sigma^{-1}, \quad \Sigma = R \cdot S \cdot R^{-1}.$$

Nevertheless we prefer to describe this proof subsequently as it was originally obtained.

By using analogous conversations one can prove the following theorem.

Theorem 2. *Let M^{13}_p be the Bazaikin space of the form $T^1_p \backslash U(5)/Sp(2) \times T_{(0)}$. Then the fixed-point set, of involution $\sigma : M^{13}_p \to M^{13}_p$, which contains the point $E = T^1_p \cdot 1 \cdot Sp(2) \times T_{(0)}$, is diffeomorphic to the space*

$$W^7_p = \begin{pmatrix} z^p & 0 & 0 \\ 0 & z^p & 0 \\ 0 & 0 & z \end{pmatrix} \backslash U(3) / \begin{pmatrix} \bar{w} & 0 & 0 \\ 0 & \bar{w} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad |z| = |w| = 1.$$

One can easy see that W^7_p is a totally geodesic submanifold of M^{13}_p. The spaces M^{13}_p are nonhomogeneous for $p \geq 2$ and thus the problem of comparing pinchings of W^7_p and M^{13}_p is not reduced to local computations as it was done in the proof of Theorem 1.

Remark. These spaces W^7_p are not presented directly in this form in [E1,E2] and probably some of these examples were not known before. We notice that they are also of the biquotient form which was introduced by Gromoll and Meyer ([GM]). As they are totally geodesic submanifolds of positively curved space they have positive sectional curvature.

The following question looks interesting.

Question. *Does there exist a correspondence of 7-dimensional Aloff-Wallach and Eschenburg spaces to 13-dimensional Berger and Bazaikin spaces which is realized by totally geodesic embeddings ?*

If such correspondence exists only for some subfamilies what are these subfamilies ?

If such correspondence exists is it realized by pinching-essential embeddings (i.e., embeddings with the same pinching constants of manifolds and submanifolds) as in the case of Theorem 1 ?

Let consider the topological properties of manifolds W^7 and B^{13}.

Put

$$\hat{G}(\approx SU(2)) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \subset SU(3), \quad A \in SU(2).$$

One can see that $SU(3)/T_{1,1} = W^7$ and $SU(3)/\hat{G} = S^5$. The group $T_{1,1}$ acts on $S^5 = \{z_1^2 + z_2^2 + z_3^2 = 1 | z_i \in \mathbf{C}\}$ by multiplications :

$$(z_1, z_2, z_3) \to (\lambda^{-2} z_1, \lambda^{-2} z_2, \lambda^{-2} z_3)$$

where $diag(\lambda, \lambda, \lambda^{-2}) \in T_{1,1}$. Moreover the actions of $T_{1,1}$ and $\hat{G}$, on $SU(3)$, commute.

Let us consider the fiber bundle

$$SU(3) \to \mathbf{C}P^2. \tag{5}$$

Its fiber is diffeomorphic to $U(2)$ which one can represent in the following manner. Put

$$\hat{Q} = SU(2) \times (\mathbf{R}/2\pi\mathbf{Z}).$$

We denote by $\hat{Q}_1$ the factor-space of $\hat{Q}$ under the following $\mathbf{Z}_2$-action:

$$(X, t) \to (-X, t + \pi), \quad X \in SU(2).$$

This factor-space is diffeomorphic to the fiber of bundle (5). It is fibered over S^1 in the usual manner

$$(X, t) \to t \in S^1 = \mathbf{R}/\pi\mathbf{Z}.$$

In these terms the action of $T_{1,1}$ on fiber bundle (5) has the form

$$(X, t) \to (\exp(\sqrt{-1}\pi\phi) \cdot X, t + \phi),$$

$$diag(\exp(\sqrt{-1}\pi\phi), \exp(\sqrt{-1}\pi\phi), \exp(-2\sqrt{-1}\pi\phi)) \in T_{1,1}.$$

Now one can derive that

1) $S^5/T_{1,1} = SU(3)/\hat{G} \times T_{1,1} = \mathbf{C}P^2$;

2) these actions generate a fiber bundle

$$W^7 = SU(3)/T_{1,1} \xrightarrow{\mathbf{R}P^3} \mathbf{C}P^2; \tag{6}$$

3) it follows from computations of cohomology groups of W^7 (see [E1]) that the transgression d_4 in the spectral sequence of fiber bundle (6) is given by

$$d_4 : E_4^{0,3} = \mathbf{Z} \xrightarrow{\times 3} E_4^{4,0} = \mathbf{Z}, \tag{7}$$

and

$$H^4(W^7) = \mathbf{Z}_3. \tag{8}$$

Put

$$\bar{G}(\approx SU(4)) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \subset SU(5), \quad A \in SU(4).$$

One can see that $SU(5)/Sp(2) \times T^1 = B^{13}$ and $SU(5)/\bar{G} = S^9$. The group T^1 acts on $S^9 = \{z_1^2 + \cdots + z_5^2 = 1 | z_i \in \mathbf{C}\}$ by multiplications :

$$(z_1, \ldots, z_5) \to (\lambda^{-4} z_1, \ldots, \lambda^{-4} z_5)$$

where $diag(\lambda, \lambda, \lambda, \lambda, \lambda^{-4}) \in T^1$. Moreover the actions of T^1 and $\bar{G}$, on $SU(5)$, commute.

Let us consider the fiber bundle

$$SU(5)/Sp(2) \to \mathbf{C}P^4. \tag{9}$$

Put

$$\bar{Q} = SU(4)/Sp(2) \times (\mathbf{R}/\pi\mathbf{Z}).$$

We denote by $\bar{Q}_1$ the factor-space of $\bar{Q}$ under the following $\mathbf{Z}_2$-action:

$$(X, t) \to (\sqrt{-1}X, t + \frac{\pi}{2}), \quad X \in SU(4)/Sp(2).$$

This factor-space is diffeomorphic to the fiber of bundle (9). It is fibered over S^1 in the usual manner

$$(X, t) \to t \in S^1 = \mathbf{R}/\frac{\pi}{2}\mathbf{Z}.$$

In these terms the action of T^1 on fiber bundle (9) has the form

$$(X, t) \to (\exp(\sqrt{-1}\pi\phi) \cdot X, t + \phi),$$
$$diag(\exp(\sqrt{-1}\pi\phi), \ldots, \exp(\sqrt{-1}\pi\phi), \exp(-4\sqrt{-1}\pi\phi)) \in T^1.$$

Now one derive that

1) $S^5/T^1 = SU(5)/\bar{G} \times T^1 = \mathbf{C}P^4$;

2) these actions generate a fiber bundle

$$B^{13} \xrightarrow{\mathbf{R}P^5} \mathbf{C}P^4; \tag{10}$$

3) it follows from computations of cohomology groups of B^{13} (see [Ba]) that the transgression d_6 in the spectral sequence of fiber bundle (10) is given by

$$d_6 : E_6^{0,5} = \mathbf{Z} \xrightarrow{\times 5} E_6^{6,0} = \mathbf{Z}. \tag{11}$$

and

$$H^6(B^{13}) = \mathbf{Z}_5. \tag{12}$$

The similarity of formulas (5-8) for W^7 and formulas (9-12) for B^{13} and Theorem 1 give us a reason to pose the following question.

Question. *Is it true that for every positive integer k there exist a space Γ_k such that*
1) there exists a fiber bundle

$$\Gamma_k \overset{\mathbf{R}P^{2k+1}}{\longrightarrow} \mathbf{C}P^{2k}; \tag{13}$$

2) the transgression d_{2k+2} in the spectral sequence of (13) is given by

$$d_{2k+2} : H^{2k+1}(S^{2k+1}) \overset{\times(2k+1)}{\longrightarrow} H^{2k+2}(\mathbf{C}P^{2k}), \tag{14}$$

and

$$H^{2k+2}(\Gamma_k) = \mathbf{Z}_{2k+1} \quad ; \tag{15}$$

3) a manifold Γ_k has positive sectional curvature ;
4) $\Gamma_1 = W^7$ and $\Gamma_2 = B^{13}$?

One can pose more rigorous conjecture by adding
5) the spaces Γ_k form a tower

$$\Gamma_1 \to \Gamma_2 \to \cdots \to \Gamma_n \to \Gamma_{n+1} \to \dots$$

of pinching-essential totally geodesic embeddings and, thus, the pinching constants of Γ_k are equal to $\frac{16}{29\cdot 37}$.

If such tower exists one can expect that its properties are similar to the properties of $\mathbf{C}P^n$- or $\mathbf{H}P^n$-towers.

Final remarks.

1) Let consider topological properties of the spaces W^7_p and M^{13}_p.

Since a left-side multiplication transforms orbits under right-side action into orbits under this action, the group $\tilde{T}_p = diag(z^p, z^p, z)$ acts on the space $U(3)/SU(2)\cdot diag(\bar{w}, \bar{w}, 1)$ $= S^5$ (where $|z| = |w| = 1$) and the group T^1_p acts on the space $U(5)/SU(4) \cdot T_{(0)} = S^9$.

These actions are not free.

One can see that the elements $diag(z^p, z^p, z)$ for $z^p = 1$ have nontrivial fixed point sets and other elements of $\tilde{T}_p$ act freely. These fixed point sets are the same for all p-roots of the unit and they are diffeomorphic to the 3-dimensional equator sphere in S^5. Let consider the $\mathbf{Z}_p$-action on S^5 which is given by the subgroup of $\tilde{T}_p$ formed by elements with $z^p = 1$. The factor-space of $U(3)/SU(2) \cdot diag(\bar{w}, \bar{w}, 1)$ under this $\mathbf{Z}_p$-action is diffeomorphic to S^5 (one can consider S^5 as the cyclic p-covering of S^5 ramified at the 3-dimensional equator sphere). The factor-group $\tilde{T}_p/\mathbf{Z}_p$ acts freely on this factor-space and for every p we obtain the mapping

$$W^7_p \longrightarrow \begin{pmatrix} z^p & 0 & 0 \\ 0 & z^p & 0 \\ 0 & 0 & z \end{pmatrix} \backslash U(3)/SU(2) \cdot \begin{pmatrix} \bar{w} & 0 & 0 \\ 0 & \bar{w} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{C}P^2.$$

It is almost evident that this mapping is the $\mathbf{R}P^3$-bundle but the strong proof needs an additional work.

Recently Ya. V. Bazaikin answering to our question computed that the order of $H^4(W_p^7)$ is equal to $r_p = (4p-1)$.

The 13-dimensional case is absolutely analogous to the 7-dimensional case and for every p we obtain the mapping

$$M_p^{13} \longrightarrow T_p^1 \backslash U(5)/SU(4)\cdot T_{(0)} = \mathbf{C}P^4.$$

This mapping also ought to be an $\mathbf{R}P^5$-bundle.

It was computed in [Ba] that the order of $H^6(M_p^{13})$ is equal to $s_p = (8p^2-4p+1)$.

One can notice that the very interesting formula

$$s_p = \frac{r_p^2+1}{2}$$

holds. For $p=1$ it coincides with (8) and (12).

One can also ask how to extend these embeddings $W_p^7 \to M_p^{13}$ to towers.

2) When we discussed the results of this paper with K. Grove he asked about the existence of such embeddings for the even-dimensional Wallach spaces ([W]). The existence of topological embeddings of these flag spaces is evident. We can give a simplest example of an involution

$$A \to \hat{S}\cdot A\cdot \hat{S}^{-1},$$

$$A = \begin{pmatrix} \sqrt{-1}I_3 & 0 \\ 0 & -\sqrt{-1}I_3 \end{pmatrix}.$$

This involution generates an involution on the space $Sp(3)/(Sp(1)\times Sp(1)\times Sp(1))$. The component of the fixed point set, which contains the orbit of the unit, is diffeomorphic to the space $SU(3)/T^2$ where T^2 is the maximal torus. Thus we obtain the totally geodesic embedding $SU(3)/T^2 \to Sp(3)/(Sp(1)\times Sp(1)\times Sp(1))$.

We remind that in [V] it was proved that homogeneous metrics on the even-dimensional Wallach spaces have the same maximal pinching which is equal to $1/64$. Thus one can expect that this embedding is pinching-essential and that for the other pair of the Wallach spaces $(F_4/Spin(8), Sp(3)/(Sp(1)\times Sp(1)\times Sp(1)))$ such embedding also exists.

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